THE FIFTH COEFFICIENT APPROXIMATION OF THE INVERSE STRONGLY CONVEX FUNCTION

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ABSTRACT

This paper discusses the fifth coefficient approximation of the inverse strongly convex function. Strongly convex function is a subclass of convex function. Those functions are included as univalent functions. Using corresponding lemmas, we give sharp limits for the fifth coefficient of the inverse strongly convex function. The limit is sharp if the value of the approximation has the same value as the limit. We verify that the limit of the fifth coefficient of the inverse strongly convex function differs from that of the strongly convex function in some interval but still have the same value in a point. Besides, we also explain that the sharp limit of the fifth inverse coefficient is less than or equal to one.

Keywords: Fifth coefficient, inverse function, strongly convex function, univalent function.

INTRODUCTION

Mathematics studies numbers, the formulation of calculations, measurements, two- and three-dimensional graphs, and related topics. The following mathematics study is pure mathematics, divided into several branches. Generally, pure mathematics is divided into four scientific branches: arithmetic, algebra, analysis, and geometry (Nasip, 2023; Ndemo & Mtetwa, 2021; Yadav, 2017). Arithmetic studies the basic operations used in mathematics, such as addition, subtraction, multiplication, and division. When these operations operate on two or more numbers whose values are unknown, these numbers will be written as a variable, making it a topic of discussion in algebra. The analysis studies the properties of the rate of change of unknown variables in algebra. In geometry, the shape and size of a graph are discussed. The fundamental elements of geometry are points, lines, angles, and planes.

This article discusses the approximation of the coefficients of the five inverse strongly convex functions, especially regarding calculating sharp limits and in which intervals these limits apply. Because there is an analysis of applicable intervals, this article tends to fall into the scientific analysis field. The analysis study has branches, such as calculus, real analysis, and complex analysis (Chihoro, 2022; Cichoń et al., 2023; Liang & Su, 2023; Schirn, 2023). The strongly convex function in this article is one of the topics discussed in complex analysis.

Mathematics does not only have pure branches but also applied branches because it can develop according to the times. This means that mathematics can be a tool in various fields, such as technology, health, and economics. In technology, algebra is the basis for programming algorithms (Humble, 2023). In health, mathematics forms a mathematical model with a fixed point as a solution. This fixed point can determine patient treatment (Fleck & Cassandras, 2017; Lobato et al., 2016; Pang et al., 2016). In economics, mathematics is a calculation tool with specific methods to minimize losses and maximize profits. The convex function itself can be used to determine local extreme points. In a mathematical model, this extreme point can represent a loss. Through the convex function, losses can be calculated to a minimum. This also applies strongly to convex functions (Niezgoda, 2022; Sarpong et al., 2018). However, this article focuses more on strongly convex functions analytically.

In complex analysis, the main topic is complex numbers. Complex numbers have a real part and an imaginary part. Complex numbers are usually written as $z = x + iy$, where x is the real part, and iy is the imaginary part. The real part is usually written as $\text{Re}(z)$, and the imaginary part is $\text{Im}(z)$. The distance from the coordinate point $(0,0)$ to a point z is called the modulus, while the angle formed between the x-axis and point z is called the argument and is usually written as $\arg(z)$ (John, 2003; Reséndis O et al., 2023; Sarason, 2007). Some theories studied in complex analysis include analytic and univalent functions. These two theories later became the basis for the emergence of strongly convex functions.

For example, $f(z)$ is a function on the complex plane with domain $U \subseteq \mathbb{C}$ and $z_0 \in \text{Int}(U)$. The f function is analytic at z_0 if f has derivatives for each $z \in N_\epsilon(z_0) \subseteq U$ (John, 2003; Sarason, 2007). Analytical functions are a condition for a function to be a univalent function. For example, S is a class of analytic functions. The analytical function f in the S class is defined on a disk $\mathbb{D} = \{z: |z| < 1\}$. The $f \in S$ function is called a univalent function if f is an injective function. The f univalent function can be written as a Taylor series expansion as in Equation (1) (Riaz et al., 2022).

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{1}$$

The f function, as in Equation 1, can be written in another form called a normalized univalent function. The univalent f function is normalized if $z_0 = 0$, $a_0 = 0$, $a_1 = 1$, and $a_n \in \mathbb{C}$ (Marjono & Thomas, 2019; Cho et al., 2018). These values are substituted in Equation (1) to obtain Equation (2).

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

As time passes, the theory regarding univalent functions develops, so new subclasses emerge. Some subclasses of univalent functions include close-to-convex function ($\mathcal{K}$) (Trabka-Więcław & Zaprawa, 2018), λ-spiral-like function ($\mathcal{S}_\lambda$) (Srivastava et al., 2019), starlike function ($\mathcal{S}^*$), convex function ($\mathcal{C}$) (Thomas & Verma, 2017), a function that has been verified to be a univalent function called the Bazilevič function ($\mathcal{B}$) (Kwon et al., 2020). These subclasses of univalent functions have the following relation: C⊂S^*⊂S_λ⊂K⊂B. $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}_\lambda \subset \mathcal{K} \subset \mathcal{B}$.

Further research was carried out to obtain coefficient approximations of the subclasses of univalent functions. The verification of the approximate value must have sharp limits so it is considered valid. The example is $p \in \mathcal{P}$, where $\mathcal{P}$ is a class function that meets $\text{Re}\, p(z) > 0$ for $z \in \mathbb{D} = \{z: |z| < 1\}$ as in Equation (3).

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{3}$$

If there are c_1, c_2, ..., c_{n-1} values as in Equation (3) so that a_n in Equation (2) has the same value as the limit of the $|a_n|$ coefficient approximation, then the limit of the approximation is sharp (Marjono & Thomas, 2019; Sim & Thomas, 2020; Thomas & Verma, 2017). In other words, the limit is still valid because it can be reached through approximation. Next, lemmas were born as tools to prove approximate values so that these values have sharp limits. These lemmas can be applied to univalent functions and their subclasses.

The development of science in the field of univalent functions is continuing. The strongly starlike function is introduced as a new function, a subclass of the starlike function. The approximation of the first to third coefficients of the strongly starlike function is verified first and has sharp limits (Cho et al., 2021; Ullah et al., 2021). Further research verifies the approximation of the fourth and fifth coefficients of the strongly starlike function. The last two coefficients are more challenging to verify because more lemmas are needed. However, re-verifying the coefficients of the four strongly starlike functions is carried out to eliminate errors in the previous verification (Lecko & Sim, 2017). The result of the coefficient approximation is a value with sharp limits. Besides verifying the coefficients of the strongly starlike function, the verification is also carried out on the inverse strongly starlike function coefficients. Although verifying the inverse function is not easy, results with sharp limits for approximating the second to fourth coefficients of the strongly starlike inverse function are obtained (Nunokawa & Sokól, 2014; Saliu et al., 2023).

Apart from the starlike subclass, the convex subclass is also experiencing development. A f function, as in Equation (2), is called a convex function if it maps $\mathbb{D}$ to a convex set. Geometrically, a convex set in a complex plane has oval, circular, or square characteristics. If a straight line is drawn between two points in a convex set, the line must still be in the convex set from which the two points were drawn. It will determine the nature of the shape of the convex set geometrically. Mapping a function to a convex set is equivalent to certain conditions. A f function is called a convex function if and only if Re $\left(1+\frac{zf''(z)}{f'(z)}\right)>0,$ for $z\in\mathbb{D}$ (Sim & Thomas, 2020). One of the convex subclass developments is the emergence of strongly convex functions. A f function as in Equation (2) with $0<\beta\leq 1$ is considered a strongly convex function if it meets $\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\pi\beta}{2}$, for $z\in\mathbb{D}$ (Arjevani et al., 2016; Thomas & Verma, 2017; Daniswara et al., 2020).

Not only is the coefficient approximation of the strongly starlike function but the second to the fourth coefficient approximation of the strongly convex function subclass is also verified. The following are the results with sharp limits of this approximation with $0<\beta\leq 1$.

$$|a_2|\leq\beta,\quad |a_3|\leq\begin{cases}\dfrac{\beta}{3} & ,0<\beta\leq\dfrac{1}{3},\\ \beta^2 & ,\dfrac{1}{3}\leq\beta\leq 1,\end{cases}\quad |a_4|\leq\begin{cases}\dfrac{\beta}{6} & ,0<\beta\leq\sqrt{\dfrac{2}{17}},\\ \dfrac{\beta}{18}(1+17\beta^2) & ,\sqrt{\dfrac{2}{17}}\leq\beta\leq 1.\end{cases}$$

The inverse function coefficients of strongly convex functions will be verified in subsequent research. If f is a strongly convex function, which means it is a univalent function, then there is an inverse function on the $|\omega|<r_0(f)$ disk with a Taylor expansion as in Equation (4).

$$f^{-1}(\omega)=z+A_2\omega^2+A_3\omega^3+A_4\omega^4+A_5\omega^5+\cdots,\tag{4}$$

A_2, A_3, A_4, and so on are the coefficients of the inverse strongly convex function. Likewise, with the approximation of inverse coefficients in subclasses of univalent functions. If there are c_1, c_2, ..., c_{n-1} as in Equation (3), such that A_n as in Equation (4) has the same value as the limit of the inverse coefficient approximation of $|A_n|$, then the limit obtained from the approximation is sharp. This also means that this limit can still be achieved by approximation, so it is considered valid. Based on the research results, which state that the approximation of the coefficient of the inverse strongly convex function is less than equal to one with $2 \leq n \leq 7$, verification is carried out on $|A_2|$, $|A_3|$, and $|A_4|$ and obtained approximation results with the same limits as the limits of $|a_2|$, $|a_3|$, dan $|a_4|$ (Thomas & Verma, 2017). In recent research, the invariance of the convex function coefficients has been verified so that the Inequality obtained results from sharp boundaries (Thomas, 2022). Additionally, the $|a_5|$ approximation is verified and did obtain results with sharp limits and $\beta_0 = 0.350162$ (Daniswara et al., 2020) in Equation (5).

$$|a_5| \leq \begin{cases} \frac{1}{10}\beta & , 0 < \beta \leq \beta_0, \\ \frac{1}{45}\beta^2(38\beta^2 + 7) & , \frac{9}{22} \leq \beta \leq 1, \end{cases} \tag{5}$$

This article aims to approximate the fifth coefficient of the inverse strongly convex function $|A_5|$. This coefficient makes the f^{-1} function with the Taylor series expansion, as in Equation (4), closer to the original function. Since the approximation $|a_5|$ has a result with sharp limits, it will be verified that the approximation $|A_5|$ has a result with sharp limits. Since $|A_5|$ has a result with sharp limits, this value will be compared with the approximation $|a_5|$. It will also be explained that the approximation value is still less than equal to one.

METHOD

The method used applies complex analytical tools in the form of lemmas suitable for verification (Ravichandran & Verma, 2015; Marjono et al., 2017; Thomas, 2017; Fitri et al., 2020). First, the second to fifth coefficients of the strongly convex function are calculated. It needs to be done because these coefficients are needed to calculate the second to fifth coefficients of the inverse strongly convex function. Once calculated, the approximation with sharp limits can be verified using the appropriate lemmas, especially the fifth coefficient of the inverse strongly convex function. Verification using this lemma sometimes does not verify the entire interval. The values are written in another form so the lemma can verify approximation on other desired intervals. The following are the lemmas that will be used in the verification.

Lemma 1. If the p function with c_n coefficients is as in Equation (3), it applies

$$|c_n| \leq 2,\ n = 1,\ 2,\ 3,\ \ldots$$

Lemma 2. If the p function with c_n coefficients is as in Equation (3), it applies

$$|c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4| \leq 2.$$

Lemma 3. If the p function with c_n coefficients is as in Equation (3), it applies

$$|c_3 - (\mu_1 + 1)c_1c_2 + \mu_1 c_1^3| \leq \begin{cases} 2 & , 0 \leq \mu_1 \leq 2 \\ 2|2\mu_1 - 1| & , \text{lainnya.} \end{cases}$$

Lemma 4. If the p function with c_n coefficients is as in Equation (3), it applies

$$\left|c_2 - \frac{\mu_2 c_1^2}{2}\right| \leq \begin{cases} 2 & , 0 \leq \mu_2 \leq 2, \\ 2|\mu_2 - 1| & , \text{others.} \end{cases}$$

Lemma 5. If the p function with c_n coefficients is as in Equation (3), it applies

$$|c_4 - c_1c_3 - \mu_3 c_2^2 + \mu_3 c_1^2 c_2| \leq \begin{cases} 2 & , 0 \leq \mu_3 \leq 1, \\ 2|2\mu_3 - 1| & , \mu_3 \geq 1. \end{cases}.$$

RESULT AND DISCUSSION

This section will prove the theorem regarding the approximation value of the fifth coefficient of the inverse strongly convex function, discuss the approximation value of the fifth coefficient of the inverse strongly convex function with the fifth coefficient of the strongly convex function, and discuss that the approximation of the fifth coefficient of the inverse strongly convex function has a value with a sharp limit that is less than equal to one.

Theorem 1. If f is a strongly convex function, there is an inverse function on the disk $|\omega| < r_0(f)$, which can be written with the Taylor f^{-1} expansion s as in Equation (4), we get Equation (6).

$$|A_5| \leq \frac{1}{45}\beta^2(8 + 37\beta^2),\ \beta_1 \leq \beta \leq 1, \tag{6}$$

with $\beta_1 = 0.453565$ as the only positive real root of the equation.

$$20\beta^2 + 24\beta - 15 = 0.$$

Verification. Before the theorem is proven using lemmas, the values of the second to fifth coefficients of the strongly convex function will be calculated first. Because f is a strongly convex function, it is a univalent function. The $p,$ as in Equation (3), is also a univalent function. Because both f and p functions are univalent functions, there is a condition where $1 + \frac{zf''(z)}{f'(z)} = (1 + \sum_{n=1}^{\infty} c_n z^n)^\beta$. A simple calculation is carried out on this equation to obtain the second to fifth coefficients of the following strongly convex function.

$$\begin{aligned}
a_2 &= \frac{1}{2}c_1\beta, \\
a_3 &= \frac{1}{12}\beta\big(2c_2 + c_1^2(3\beta - 1)\big), \\
a_4 &= \frac{1}{144}\beta\big(12c_3 + 6c_1c_2(5\beta - 2) + c_1^3(17\beta^2 - 15\beta + 4)\big), \text{ dan} \\
a_5 &= \frac{1}{720}\beta(36c_4 + 18(2\beta - 1)c_2^2 + 12(7\beta - 3)c_{1_{c_3}} \\
&\qquad +12(10\beta^2 - 10\beta + 3)c_1^2c_2 + (38\beta^3 - 60\beta^2 + 37\beta - 9)c_1^4.
\end{aligned}$$

The values of the second to fifth coefficients of the strongly convex function are then used to calculate the second to fifth coefficients of the inverse strongly convex function. Because f^{-1}, as in Equation (4), is an inverse function, it applies $f\big(f^{-1}(\omega)\big) = \omega$. This equation can calculate the coefficient value of the inverse strongly convex function. A simple calculation is carried out to obtain the following values for the second to fifth coefficients of the inverse strongly convex function.

$$\begin{aligned}
A_2 &= -a_2, \\
A_3 &= -(a_3 + 2a_2A_2), \\
A_4 &= -(a_4 + 3a_3A_2 + a_2A_2^2 + 2a_2A_3), \text{ and} \\
A_5 &= -(a_5 + 4a_4A_2 + 3a_3A_2^2 + 3a_3A_3 + 2a_2A_2A_3 + 2a_2A_4)
\end{aligned}$$

The second to fifth coefficients of the inverse strongly convex function still contain the second to fifth coefficients of the strongly convex function. This form can be written in another form so that it does not contain the second to fifth coefficients of the strongly convex function. This can be done by substituting the second to fifth coefficients of the strongly convex function of A_2, A_3, A_4, and A_5. After the substitution process, the following form is obtained.

$$
\begin{aligned}
A_2 &= -\tfrac{1}{2}c_1\beta, \\
A_3 &= \tfrac{1}{12}\beta((1+3\beta)c_1^2 - 2c_2), \\
A_4 &= -\tfrac{1}{144}\beta(12c_3 - 6(2+5\beta)c_1c_2 + (4+15\beta+17\beta^2)c_1^3), \text{ and} \\
A_5 &= \tfrac{1}{720}\beta(-36c_4 + 6(3+4\beta)c_2^2 + 12(3+8\beta)c_1c_3 \\
&\quad -12(3+10\beta+10\beta^2)c_1^2c_2 + (9+38\beta+60\beta^2+37\beta^3)c_1^4). \qquad (7)
\end{aligned}
$$

After obtaining the fifth coefficient of the inverse strongly convex function f^{-1}, it will then be verified that the results obtained have sharp limits. Verification is carried out by applying lemmas. In order to apply the appropriate lemma, it is necessary to write appropriately A_5 as in Equation (7). The first form of writing can be written as follows.

$$A_5 = \frac{1}{720}\beta\xi_1,$$

with

$$
\begin{aligned}
\xi_1 = 36\phi_1 &+ 6(4\beta-3)c_2^2 + (-40\beta^2+56\beta-12)c_1c_3 \\
&+8(5\beta^2+5\beta-3)c_1\phi_2 - \tfrac{1}{45}(-37\beta^4+20\beta^3+42\beta^2-21\beta)c_1^2\phi_3 \\
&+\tfrac{1}{45}(-37\beta^4+20\beta^3+42\beta^2-21\beta)c_1^2c_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_1 &= c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4, \\
\phi_2 &= c_3 - 3c_1c_2 + 2c_1^3, \\
\phi_3 &= c_2 - \tfrac{45(-37\beta^3+20\beta^2+42\beta-21)c_1^2}{(37\beta^4-20\beta^3-42\beta^2+21\beta)}.
\end{aligned}
$$

Because $p \in \mathcal{P}$, the lemma is applied to ξ_1 if the coefficient of each term in ξ_1 is positive. In other words, the coefficients of ϕ_1, c_2^2, c_1c_3, $c_1\phi_2$, $c_1^2\phi_3$, and $c_1^2c_2$ must be positive. Conditions are determined so that the coefficients are positive. The coefficient of ϕ_1 is positive, so no additional conditions are needed for the lemma to apply. The coefficient of c_2^2 will be positive if $\beta \geq \frac{3}{4}$. The coefficient of c_1c_3 will be positive if $\beta \geq 0.26411$. The coefficient of $c_1\phi_2$ will be positive if $\beta \geq 0.421954$. The coefficient of $c_1^2\phi_3$ dan $c_1^2c_2$ will be positive if $\beta \geq 0.489171$. Because the conditions obtained are different, a new condition is determined to apply to all terms by slicing up all the specified conditions. The intersection of each term in ξ_1 is $\beta \geq \frac{3}{4}$. It means every term in ξ_1 will be positive if $\beta \geq \frac{3}{4}$.

Based on the definition of a strongly convex function, the value of β is only in the interval of $0 < \beta \leq 1$. Therefore, the interval determined for ξ_1 to be positive has a new condition: $0 < \beta \leq$

1. Therefore, this condition is sliced with $0 < \beta \leq 1$ to obtain the new condition: $\frac{3}{4} \leq \beta \leq 1$. It can be concluded that the corresponding lemmas can be applied to ξ_1 if $\frac{3}{4} \leq \beta \leq 1$. Lemma 1, Lemma 2, and Lemma 3 are applied with $\mu_1 = 2$, and Lemma 4 with μ_2, which has a value of more than one $\mu_2 = \frac{90(-37\beta^3+20\beta^2+42\beta-21)}{(37\beta^4-20\beta^3-42\beta^2+21\beta)}$ in ξ_1 so we get:

$$
\begin{aligned}
|\xi_1| &= |36\phi_1 + 6(4\beta-3)c_2^2 + (-40\beta^2+56\beta-12)c_1c_3 \\
&\quad +8(5\beta^2+5\beta-3)c_1\phi_2 - \tfrac{1}{45}(-37\beta^4+20\beta^3+42\beta^2-21\beta)c_1^2\phi_3 \\
&\quad +\tfrac{1}{45}(-37\beta^4+20\beta^3+42\beta^2-21\beta)c_1^2c_2| \\
&\leq 36|\phi_1| + 6(4\beta-3)|c_2^2| + (-40\beta^2+56\beta-12)|c_1||c_3| \\
&\quad +8(5\beta^2+5\beta-3)|c_1||\phi_2| + \tfrac{1}{45}(-37\beta^4+20\beta^3+42\beta^2-21\beta)|c_1^2||\phi_3| \\
&\quad +\tfrac{1}{45}(-37\beta^4+20\beta^3+42\beta^2-21\beta)|c_1^2||c_2| \\
&\leq 128\beta + 592\beta^3.
\end{aligned}
$$

The approximation value ξ_1 is substituted to approximate the A_5 value, and it obtained $|A_5| = \left|\frac{1}{720}\beta\xi_1\right| \leq \frac{1}{45}\beta^2(8+37\beta^2)$. It means that the limit of approximation is $\frac{1}{45}\beta^2(8+37\beta^2)$. Next, it will be verified that the limit is sharp by determining c_1, c_2, c_3, and c_4 so that Equation (7) has the value of $\frac{1}{45}\beta^2(8+37\beta^2)$. The values of $c_1 = 2$, $c_2 = 2$, $c_3 = 2$, and $c_4 = 2$ were selected and substituted in Equation (7), then $A_5 = \frac{1}{45}\beta^2(8+37\beta^2)$ was obtained. It proves that the value of the A_5 approximation using the form ξ_1 has a sharp limit.

Because the condition for every term in ξ_1 to be positive is $\frac{3}{4} \leq \beta \leq 1$, this verification only applies to that interval. Therefore, verifications on other intervals can be done by writing Equation (7) differently. The form must have conditions different from ξ_1 to cover intervals other than $\frac{3}{4} \leq \beta \leq 1$. One form of writing Equation (7) is as follows.

$$A_5 = \frac{1}{720}\beta\xi_2,$$

with

$$
\begin{aligned}
\xi_2 = 6(9-4\beta)\psi_1 &+ 6(3-4\beta)\psi_2 + 8(-5\beta^2+9\beta-3)c_1c_3 \\
&+2(20\beta^2+24\beta-15)c_1\psi_3 + \tfrac{1}{45}(-37\beta^4+20\beta^3+34\beta^2-15\beta)c_1^2c_2 \\
&-\tfrac{1}{45}(-37\beta^4+20\beta^3+34\beta^2-15\beta)c_1^2\psi_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_1 &= c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4, \\
\psi_2 &= c_4 - c_1c_3 - 2c_2^2 + 2c_1^2c_2, \\
\psi_3 &= c_3 - 3c_1c_2 + 2c_1^3,
\end{aligned}
$$

$$\psi_4 = c_2 - \frac{45(-37\beta^3+20\beta^2+34\beta-15)c_1^2}{37\beta^4-20\beta^3-34\beta^2+15\beta}.$$

The same steps in ξ_1 are used to verify ξ_2. Because $p \in \mathcal{P}$, the lemma can be applied to ξ_2 if the coefficient of each term in ξ_2 is positive. In other words, the coefficients of ψ_1, ψ_2, c_1c_3, $c_1\psi_3$, $c_1^2c_2$, and $c_1^2\psi_4$ must be positive. Conditions are determined so that the coefficients are positive. The coefficient of ψ_1 will be positive if $\beta \leq \frac{9}{4}$. The coefficient of ψ_2 will be positive if $\beta \leq \frac{3}{4}$. The coefficient of c_1c_3 will be positive if $\beta \geq 0.441742$. The coefficient of $c_1\psi_3$ will be positive if $\beta \geq \beta_1 = 0.453565$. The coefficient of $c_1^2c_2$ and $c_1^2\psi_4$ will be positive if $\beta \geq 0.417866$. All these conditions are sliced so that the coefficient of each term in ξ_2 will be positive if $0.453565 = \beta_1 \leq \beta \leq \frac{3}{4}$. Based on the definition of the strongly convex function, the applicable β value is in the interval $0 < \beta \leq 1$. Because the condition in the form of the interval $\beta_1 \leq \beta \leq \frac{3}{4}$ is a subinterval of $0 < \beta \leq 1$, it fulfills the condition for ξ_2 to be positive. It can be concluded that the corresponding lemmas can be applied to ξ_2 if $\beta_1 \leq \beta \leq \frac{3}{4}$. Lemma 1, Lemma 2, and Lemma 3 are applied with $\mu_1 = 2$, and Lemma 4 with μ_2, which has a value of more than one $\mu_2 = \frac{90(-37\beta^3+20\beta^2+42\beta-21)}{(37\beta^4-20\beta^3-42\beta^2+21\beta)}$, and Lemma 5 with $\mu_3 = 2$ in ξ_2, so we get:

$$\begin{aligned}
|\xi_2| &= |6(9-4\beta)\psi_1 + 6(3-4\beta)\psi_2 + 8(-5\beta^2+9\beta-3)c_1c_3 \\
&\quad +2(20\beta^2+24\beta-15)c_1\psi_3 + \frac{1}{45}(-37\beta^4+20\beta^3+34\beta^2-15\beta)c_1^2c_2 \\
&\quad -\frac{1}{45}(-37\beta^4+20\beta^3+34\beta^2-15\beta)c_1^2\psi_4| \\
&\leq 6(9-4\beta)|\psi_1| + 6(3-4\beta)|\psi_2| + 8(-5\beta^2+9\beta-3)|c_1||c_3| \\
&\quad +2(20\beta^2+24\beta-15)|c_1||\psi_3| + \frac{1}{45}(-37\beta^4+20\beta^3+34\beta^2 - \\
&\quad 15\beta)|c_1^2||c_2| \\
&\quad +\frac{1}{45}(-37\beta^4+20\beta^3+34\beta^2-15\beta)|c_1^2||\psi_4| \\
&\leq 128\beta + 592\beta^3.
\end{aligned}$$

The approximation value ξ_2 is substituted to approximate the A_5 value, and it obtained $|A_5| = \left|\frac{1}{720}\beta\xi_1\right| \leq \frac{1}{45}\beta^2(8+37\beta^2)$. Because $\frac{1}{45}\beta^2(8+37\beta^2)$ is verified to be a sharp limit, verifying the approximation value of A_5 using ξ_2 form produces results with a sharp limit. Because the condition for every term in ξ_2 to be positive is $\beta_1 \leq \beta \leq \frac{3}{4}$, this verification only applies to the interval of $\beta_1 \leq \beta \leq \frac{3}{4}$. Based on verifying the approximation values of the two other forms of Equation (7) previously, it can be concluded that the approximation coefficients of the five inverse strongly convex functions have values with sharp limits in the interval of $\beta_1 \leq \beta \leq \frac{3}{4}$.

These results prove Theorem 1 that the approximate value of the coefficients of the five inverse strongly convex functions is $|A_5| \leq \frac{1}{45}\beta^2(8+37\beta^2)$. This verification applies to a certain β interval, $0.453565 = \beta_1 \leq \beta \leq 1$. This approximation value is a sharp limit. Therefore, there is a strongly convex inverse function as in Equation (4) with $|A_5| = \frac{1}{45}\beta^2(8+37\beta^2)$. This can happen if $c_1 = 2$, $c_2 = 2$, $c_3 = 2$, and $c_4 = 2$ are chosen. Apart from that, through this verification, it can be interpreted that there is an inverse function as in Equation (4) with $|A_5|$ as in Equation (6) so that the function is close to the original function.

Compared with the approximate value $|a_5|$ as in Equation (5), the approximate value $|A_5|$ in $\beta_1 \leq \beta < 1$ has different values. This can be proven simply by substituting the β value with $\beta_1 \leq \beta < 1$ in each approximation value; then, an approximation value with different limits will be obtained. However, if $\beta = 1$ is chosen, an approximation value with the same limits will be obtained. This is slightly different from the approximation values of the lower coefficients, $|a_2|$, $|a_3|$, and $|a_4|$, which have the same limits as $|A_2|$, $|A_3|$, dan $|A_4|$ (Thomas & Verma, 2017; Daniswara et al., 2020).

In Inequality (6), the β value is applicable only in the $\beta_1 \leq \beta \leq 1$ interval. The applicable β values are substituted at the sharp limit, $\frac{1}{45}\beta^2(8 + 37\beta^2)$. First, $\beta = 1$ is substituted so that the limit has a value of one. Next, the value of $\beta_1 \leq \beta < 1$ is substituted for the sharp limit. Then, the β value value will be squared so that if the substituted β value is less than one, we get an even smaller value. It means that the approximation of the fifth coefficient of the inverse strongly convex function has sharp limits with a value less than equal to one (Thomas & Verma, 2017).

CONCLUSION

The approximation value of the fifth coefficient of the inverse strongly convex function is verified to have sharp limits. It means there are conditions where the approximation value is within that limit. The approximation value of this coefficient is different from the approximation value of the fifth coefficient of the strongly convex function but is still less than equal to one. Although this study has verified that the approximation value of the five coefficients of the inverse strongly convex function is a sharp limit, the entire interval, as in the definition of the strongly convex function, has not been proven. Further studies can be carried out to complete the evidence on intervals that have not been proven.

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